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Unsteady Lagally theorem for multipoles and deformable bodies

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1. Statement of the problem

The Lagally theorem for unsteady flow expresses the forces and moments acting on a rigid body moving in an inviscid and incompressible fluid in terms of the singularities of the analytically continued flow within the body. Previous generalizations of the Lagally theorem, originally given by Lagally (1922) for steady flows, are due to Cummins (1957) and Landweber & Yih (1956), who consider the effect of flow unsteadiness on the forces and moments. In these, the system of image singularities within the body was assumed to consist of isolated or continuous (surface or volume) distributions of sources and doublets. A further extension of Lagally's theorem is due to Landweber (1967), who derived expressions for the *steady* forces and moments acting on a rigid body generated by isolated or a continuous distribution of multipoles. The purpose of the present paper is to generalize the Lagally theorem so as to include the effects of multipoles in unsteady flow, and deformability of the body, as well as to present a briefer derivation of the resulting formulae. Two examples, illustrating the application of the force and moment formulae, will be presented.

We shall employ a rectangular Cartesian co-ordinate system x_i ($i = 1, 2, 3$), or alternatively x, y, z, attached to the body, and denote the position vector of the point (x_1, x_2, x_3) by $\mathbf{r}$. The components of the velocity vector $\mathbf{V}$ of the origin of co-ordinates will be denoted by (V_1, V_2, V_3) and the components of the angular velocity of the body $\boldsymbol{\omega}$ by $(\omega_1, \omega_2, \omega_3)$. The surface of the deformable body will be denoted by $\mathscr{S}$ and the volume by $\mathscr{V}$, where both $\mathscr{S}$ and $\mathscr{V}$ are, in general, time-dependent. The instantaneous direction of the outward normal to $\mathscr{S}$ will be indicated by the unit normal vector $\mathbf{n}$, with components $n_i = \partial x_i/\partial n$. Here n denotes distance measured from $\mathscr{S}$ along the normal, positive outward.

We shall assume that the fluid is inviscid and incompressible, the flow irrotational, and express the velocity potential as

$$\Phi = \sum_{i=1}^{3} (V_i\phi_i + \omega_i\phi_{i+3}) + \phi_0 + \phi_d, \quad \mathbf{v} = \nabla\Phi, \tag{1}$$

where $\mathbf{v}$ denotes the velocity at a point of the fluid, the unit potentials ϕ_i and ϕ_{i+3} represent flows induced by the motion of the undeformed body when all boundaries and external flow-producing mechanisms are at rest, ϕ_0 is due to the presence and motion of the latter when the undeformed body is at rest, and ϕ_d is the additional potential which is associated with the deformation of the body. The kinematic boundary condition at a point of $\mathscr{S}$ can then be expressed in the alternative forms

$$\mathbf{v}.\mathbf{n} = (\mathbf{V} + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{V}_d).\mathbf{n} \tag{2}$$

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or

$$\frac{\partial \phi_i}{\partial n} = n_i, \quad \frac{\partial \phi_{i+3}}{\partial n} = (\mathbf{r} \times \mathbf{n})_i, \quad \frac{\partial \phi_0}{\partial n} = 0, \quad \frac{\partial \phi_d}{\partial n} = \mathbf{V}_d . \mathbf{n}, \tag{3}$$

where $\mathbf{V}_d$ is the velocity of the deforming surface relative to the co-ordinate system x_i which is attached to the body. If we denote the equation of the body surface by $f(\mathbf{r}, t) = 0$, where t denotes time, we may derive the following expression for the deformation velocity $\mathbf{V}_d$:

$$\mathbf{n} . \mathbf{V}_d = \frac{-1}{|\nabla f|} \frac{\partial f}{\partial t}. \tag{4}$$

Relative to the moving co-ordinate system, the Bernoulli equation yields the pressure p in the form (Lamb 1932, p. 20)

$$\frac{p}{\rho} = -\frac{\partial \Phi}{\partial t'} - \tfrac{1}{2}\mathbf{v} . \mathbf{v} + \mathbf{v} . \mathbf{V} + \mathbf{v} . \boldsymbol{\omega} \times \mathbf{r}, \tag{5}$$

where ρ is the mass density of the fluid. The time variable relative to the *moving* co-ordinate system t' is related to the 'absolute' time variable t relative to a *fixed* co-ordinate system so that, for any differentiable vector $\mathbf{Q}(\mathbf{r}, t)$,

$$\frac{d\mathbf{Q}}{dt} = \frac{d\mathbf{Q}}{dt'} + \boldsymbol{\omega} \times \mathbf{Q}. \tag{6}$$

The force $\mathbf{F}$ and moment $\mathbf{M}$ acting on the body will be determined from the basic formulae

$$\mathbf{F} = -\int_{\mathscr{S}} p\mathbf{n}\, d\mathscr{S}, \quad \mathbf{M} = -\int_{\mathscr{S}} p\mathbf{r} \times \mathbf{n}\, d\mathscr{S}, \tag{7}$$

where the pressure is given by (5). We shall show that both $\mathbf{F}$ and $\mathbf{M}$ are expressible in terms of the system of singularities within the body associated with the external flow and the motion of the body surface. The internal flow will also be assumed to be irrotational and the internal singularities will at first be assumed to be isolated multipoles of order $q = \alpha + \beta + \gamma$ and strength P_{kq}, situated at a point $\mathbf{r}_s$, with potential

$$\phi_{kq} = -P_{kq} D_s^q \left(\frac{1}{R}\right), \quad D_s^q = \frac{\partial^q}{\partial x_s^\alpha\, \partial y_s^\beta\, \partial z_s^\gamma}, \quad R^2 = (x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2, \tag{8}$$

where $k = d$, 0, 1, 2, 3, 4, 5, 6 indicates that the singularity is associated with ϕ_d, ϕ_0, ϕ_i, or ϕ_{i+3} ($i = 1, 2, 3$) respectively. These include the source and doublet potentials as those of multipoles of order 0 and 1, respectively. In the neighbourhood of a multipole we shall require the definitions

$$\phi_k = \phi_k' + \phi_{kq}, \quad \mathbf{v}_k = \mathbf{v}_k' + \mathbf{v}_{kq}, \tag{9}$$

where both ϕ_k' and $\mathbf{v}_k'$ are regular at the location of the multipole.

We shall derive the following expressions for the Lagally force

$$\frac{1}{\rho}\mathbf{F} = \frac{d}{dt}\left[\mathbf{V}_c \mathscr{V} - 4\pi \sum_s P_q D_s^q(\mathbf{r})_s + \frac{d}{dt}(\mathbf{r}_c \mathscr{V})\right] - 4\pi \sum_s P_q D_s^q(\mathbf{v}')_s \tag{10}$$

and for the Lagally moment about the origin

$$\frac{1}{\rho}\mathbf{M}_j = \left[\mathbf{r}_c\times\frac{d}{dt}(\mathbf{V}\mathscr{V})\right]_j - \frac{d}{dt}\Big\{\omega_i A_{3+j,3+i} + 4\pi\sum_s [P_{3+j,q'}\, D_s^{q'}(\mathbf{V}.\mathbf{r}-\phi_0')_s + P_{0q} D_s^q(\phi'_{3+j})_s] + \int_{\mathscr{S}} \phi_{3+j}\mathbf{V}_d.\mathbf{n}\, d\mathscr{S}\Big\} - 4\pi\sum_s P_q D_s^q[\mathbf{r}\times(\mathbf{v}'-\mathbf{V})]_{js} + \left[\mathbf{V}\times\frac{d}{dt}(\mathbf{r}_c\mathscr{V})\right]_j. \quad (11)$$

Here $\mathbf{V}_c$ is the velocity of the centroid of the volume $\mathscr{V}$ of the body and $\mathbf{r}_c$ its position vector, $\mathbf{V}_d$ denotes the velocity of a point of the deformable surface $\mathscr{S}$ where $\mathbf{V}_d = 0$ corresponds to the case of a rigid surface; $\sum_s$ indicates that the summation extends over all singularities and $(\;)_s$ that the quantity in parentheses is evaluated at the point (x_s, y_s, z_s). The multipole strength is denoted by P_q, with the index k omitted when it is associated with the total velocity $\mathbf{v}$; subscript j in (11) denotes the jth component of a vector;

$$A_{ij} = -\int_{\mathscr{S}} \phi_i(\partial\phi_j/\partial n)\, d\mathscr{S} = A_{ji}$$

are the instantaneous added-mass coefficients of the body; q, q' indicate that, in general, multipoles of various orders are present.

2. Some transformations

Two basic transformations occur repeatedly in the derivation of the generalized Lagally theorem. These involve a pair of vectors $\mathbf{u}$ and $\mathbf{v}$, both regular and solenoidal, and $\mathbf{v}$ also irrotational within $\mathscr{V}$, except at a finite number of points at which multipoles are present. Then we have the first transformation

$$\int_{\mathscr{S}} (\mathbf{v}.\mathbf{un}-\mathbf{vu}.\mathbf{n})\, d\mathscr{S} = \int_{\mathscr{V}} (\nabla\mathbf{u}).\mathbf{v}\, d\mathscr{V} - \frac{8\pi}{3}\sum_s P_{uq} D_s^q(\mathbf{v}')_s, \quad (12)$$

where P_{uq} is the strength of a multipole of order q associated with $\mathbf{u}$.

To prove (12) we apply the Gauss transformation to the volume $\mathscr{V}'$, bounded externally by $\mathscr{S}$ and internally by surfaces $\mathscr{S}_0$ bounding small regions about the singularities of $\mathbf{v}$ and $\mathbf{u}$, taking the positive sense of $\mathbf{n}$ on $\mathscr{S}_0$ into the region $\mathscr{V}'$. Then we have

$$\int_{\mathscr{S}} (\mathbf{v}.\mathbf{un}-\mathbf{vu}.\mathbf{n})\, d\mathscr{S} = \sum_s \int_{\mathscr{S}_0} (\mathbf{v}\times\mathbf{n})\times\mathbf{u}\, d\mathscr{S}_0 + \int_{\mathscr{V}'} (\nabla\mathbf{u}).\mathbf{v}\, d\mathscr{V}' \quad (13)$$

since $\mathbf{v}.\mathbf{un}-\mathbf{vu}.\mathbf{n} = (\mathbf{v}\times\mathbf{n})\times\mathbf{u}$ and both $\nabla\times\mathbf{v}$ and $\nabla.\mathbf{u}$ vanish in $\mathscr{V}'$. Now put

$$(\mathbf{v}\times\mathbf{n})\times\mathbf{u} = [(\mathbf{v}'+\mathbf{v}_{q'})\times\mathbf{n}]\times(\mathbf{u}'+\mathbf{u}_q) = (\mathbf{v}'\times\mathbf{n})\times\mathbf{u}' + (\mathbf{v}'\times\mathbf{n})\times\mathbf{u}_q + (\mathbf{v}_{q'}\times\mathbf{n})\times\mathbf{u}. \quad (14)$$

Since both $\mathbf{u}'$ and $\mathbf{v}'$ are regular in the region bounded by $\mathscr{S}_0$, we have

$$\lim_{\mathscr{S}_0\to 0}\int_{\mathscr{S}_0} (\mathbf{v}'\times\mathbf{n})\times\mathbf{u}'\, d\mathscr{S}_0 = 0. \quad (15)$$

The second term in the right-hand side of (14) yields

$$\int_{\mathscr{S}_0} (\mathbf{v}'\times\mathbf{n})\times\mathbf{u}_q\, d\mathscr{S}_0 = P_{uq} D_s^q \int_{\mathscr{S}_0} (\mathbf{v}'\times\mathbf{n})\times(\mathbf{R}/R^3)\, d\mathscr{S}_0, \quad (16)$$

where $\mathbf{R}$ is the position vector of a point on $\mathscr{S}_0$ relative to (x_s, y_s, z_s) and

$$\mathbf{u}_q = P_{uq} D_s^q(\mathbf{R}/R^3).$$

At this stage we may select $\mathscr{S}_0$ to be a sphere of radius R_0 and volume $\mathscr{V}_0$ about the singularity (x_s, y_s, z_s) as centre. Here $\mathbf{R}/R_0 = \mathbf{n}$ and we obtain

$$\int_{\mathscr{S}_0} (\mathbf{v}' \times \mathbf{n}) \times \frac{\mathbf{R}}{R_0^3} d\mathscr{S}_0 = \frac{1}{R_0^2} \int_{\mathscr{S}_0} (\mathbf{v}'.\mathbf{nn} - \mathbf{v}') \, d\mathscr{S}_0. \tag{17}$$

Since $\nabla . \mathbf{v}' = 0$ in $\mathscr{V}_0$, we also have

$$\frac{1}{R_0^2} \int_{\mathscr{S}_0} \mathbf{v}'.\mathbf{nn} \, d\mathscr{S}_0 = \frac{1}{R_0^3} \int_{\mathscr{V}_0} \mathbf{v}'.\nabla \mathbf{R} \, d\mathscr{V}_0 = \frac{1}{R_0^3} \int_{\mathscr{V}_0} \mathbf{v}' d\mathscr{V}_0 \to \frac{4\pi}{3} (\mathbf{v}')_s. \tag{18}$$

Hence, as $R_0 \to 0$, (17) becomes

$$\int_{\mathscr{S}_0} (\mathbf{v}' \times \mathbf{n}) \times \frac{\mathbf{R}}{R_0^3} d\mathscr{S}_0 = \frac{1}{R_0^3} \int_{\mathscr{V}_0} \mathbf{v}' d\mathscr{V}_0 - \frac{1}{R_0^2} \int_{\mathscr{S}_0} \mathbf{v}' d\mathscr{S}_0 \to (\tfrac{4}{3}\pi - 4\pi)(\mathbf{v}')_s = -\frac{8\pi}{3} (\mathbf{v}')_s$$

and

$$\int_{\mathscr{S}_0} (\mathbf{v}' \times \mathbf{n}) \times \mathbf{u}_q \, d\mathscr{S}_0 = -\frac{8\pi}{3} P_{uq} D_s^q (\mathbf{v}')_s. \tag{19}$$

Finally, the integral of the last term in (14) vanishes because

$$\int_{\mathscr{S}_0} (\mathbf{v}_{q'} \times \mathbf{n}) \times \mathbf{u} \, d\mathscr{S}_0 = P_{vq'} D_s^{q'} \int_{\mathscr{S}_0} \frac{\mathbf{n} \times \mathbf{n}}{R_0^2} \times \mathbf{u} \, d\mathscr{S}_0 = 0. \tag{20}$$

The transformation (12) is then obtained from the results given in (13), (14), (15), (19) and (20) when the right-hand side of (19) is summed over all singularities of $\mathbf{u}$ within the volume $\mathscr{V}$, and we take the limit as $R_0 \to 0$.

We shall require three special cases of (12). When $\mathbf{u} = \mathbf{v}$, we have from (12)

$$\int_{\mathscr{S}} (\mathbf{v}.\mathbf{vn} - \mathbf{vv}.\mathbf{n}) \, dS = \int_{\mathscr{V}} \nabla \mathbf{v}.\mathbf{v} \, d\mathscr{V} - \frac{8\pi}{3} \sum_s P_q D_s^q (\mathbf{v}')_s, \tag{21}$$

where, from now on, we write P_q for P_{vq}. In addition, we may write

$$\int_{\mathscr{S}} \mathbf{v}.\mathbf{vn} \, d\mathscr{S} = \sum_s \int_{\mathscr{S}_0} \mathbf{v}.\mathbf{vn} \, d\mathscr{S} + 2 \int_{\mathscr{V}} \nabla \mathbf{v}.\mathbf{v} \, d\mathscr{V}' \to 2 \int_{\mathscr{V}} \nabla \mathbf{v}.\mathbf{v} \, d\mathscr{V} + \frac{8\pi}{3} \sum_s P_q D_s^q (\mathbf{v}')_s \tag{22}$$

since, by (18),

$$\int_{\mathscr{S}_0} \mathbf{v}.\mathbf{vn} \, d\mathscr{S}_0 = \int_{\mathscr{S}_0} (\mathbf{v}'.\mathbf{v}' + 2\mathbf{v}'.\mathbf{v}_q + \mathbf{v}_q.\mathbf{v}_q) \, \mathbf{n} \, d\mathscr{S}_0 = 2P_q D_s^q \int_{\mathscr{S}_0} \frac{\mathbf{v}'.\mathbf{nn}}{R_0^2} d\mathscr{S}_0 + (P_q D_s^q)^2 \int_{\mathscr{S}_0} \frac{\mathbf{n}}{R_0^4} d\mathscr{S}_0 \to \frac{8\pi}{3} P_q D_s^q (\mathbf{v}')_s. \tag{23}$$

Hence, eliminating the volume integral between (21) and (22), we obtain the first special transformation

$$\int_{\mathscr{S}} (\tfrac{1}{2}\mathbf{v}.\mathbf{vn} - \mathbf{vv}.\mathbf{n}) \, d\mathscr{S} = -4\pi \sum_s P_q D_s^q (\mathbf{v}')_s. \tag{24}$$

Secondly, when $\mathbf{u} = \mathbf{V}$, a constant vector, the volume integral in (12) vanishes and, since for the present case $P_{uq} = 0$, we obtain from (12)

$$\int_{\mathscr{S}} (\mathbf{v}\,.\,\mathbf{Vn} - \mathbf{vV}\,.\,\mathbf{n})\, d\mathscr{S} = 0. \tag{25}$$

Thirdly, we set $\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r}$ and express $\mathbf{v}$ as $\mathbf{v} = \nabla\Phi$, with $\Phi = \Phi' + \Phi_{q'}$ and Φ' regular within $\mathscr{S}_0$. Then we have $P_{uq} = 0$,

$$\int_{\mathscr{S}_0} \Phi\mathbf{n}\, d\mathscr{S}_0 = \int_{\mathscr{S}_0} \Phi_q \mathbf{n}\, d\mathscr{S}_0 = -P_q D_s^q \int_{\mathscr{S}_0} \frac{\mathbf{n}}{R_0}\, d\mathscr{S}_0 = 0 \tag{26}$$

and therefore, since also $\nabla(\boldsymbol{\omega} \times \mathbf{r})\,.\,\mathbf{v} = -\boldsymbol{\omega} \times \mathbf{v}$,

$$\int_{\mathscr{V}'} \nabla(\boldsymbol{\omega} \times \mathbf{r})\,.\,\mathbf{v}\, d\mathscr{V}' = -\boldsymbol{\omega} \times \int_{\mathscr{V}'} \nabla\Phi\, d\mathscr{V}' = -\boldsymbol{\omega} \times \left[\int_{\mathscr{S}} \Phi\mathbf{n}\, d\mathscr{S} - \sum_s \int_{\mathscr{S}_0} \Phi\mathbf{n}\, d\mathscr{S}_0\right]$$

$$= -\boldsymbol{\omega} \times \int_{\mathscr{S}} \Phi\mathbf{n}\, d\mathscr{S}. \tag{27}$$

Hence (12) becomes

$$\int_{\mathscr{S}} (\mathbf{v}\,.\,\boldsymbol{\omega} \times \mathbf{rn} - \mathbf{v}\boldsymbol{\omega} \times \mathbf{r}\,.\,\mathbf{n})\, d\mathscr{S} = -\boldsymbol{\omega} \times \int_{\mathscr{S}} \Phi\mathbf{n}\, d\mathscr{S}. \tag{28}$$

The second transformation is the following:

$$\int_{\mathscr{S}} \mathbf{r} \times [\mathbf{v}\,.\,\mathbf{un} - \mathbf{vu}\,.\,\mathbf{n}]\, d\mathscr{S} = \int_{\mathscr{V}} [\mathbf{r} \times \nabla\mathbf{u}\,.\,\mathbf{v} + \mathbf{v} \times \mathbf{u}]\, d\mathscr{V} - \frac{8\pi}{3} \sum_s P_{uq} D_s^q (\mathbf{r} \times \mathbf{v}')_s, \tag{29}$$

which may be proved by applying the Gauss transformation to the volume $\mathscr{V}'$, the hypothesis $\nabla \times \mathbf{v} = \nabla\,.\,\mathbf{v} = \nabla\,.\,\mathbf{u} = 0$, the identities

$$\mathbf{v} \times \nabla\,.\,\mathbf{r} = 0, \quad \mathbf{v} \times \nabla\mathbf{r}\,.\,\mathbf{u} = \mathbf{v} \times \mathbf{u}, \tag{30}$$

and, by (19),

$$\int_{\mathscr{S}_0} \mathbf{r} \times [(\mathbf{v}' \times \mathbf{n}) \times \mathbf{u}_q]\, d\mathscr{S}_0 \to (\mathbf{r})_s \times \int_{\mathscr{S}_0} (\mathbf{v}' \times \mathbf{n}) \times \mathbf{u}_q\, d\mathscr{S}_0 \to -\frac{8\pi}{3} P_{uq} D_s^q (\mathbf{r} \times \mathbf{v}')_s. \tag{31}$$

Again, we may derive some special forms of the transformation (29). When $\mathbf{u} = \mathbf{v}$, we have

$$\int_{\mathscr{S}} \mathbf{r} \times [\mathbf{v}\,.\,\mathbf{vn} - \mathbf{vv}\,.\,\mathbf{n}]\, d\mathscr{S} = \int_{\mathscr{V}} \mathbf{r} \times (\nabla\mathbf{v})\,.\,\mathbf{v}\, d\mathscr{V} - \frac{8\pi}{3} P_{vq} D_s^q (\mathbf{r} \times \mathbf{v}')_s, \tag{32}$$

but

$$\frac{1}{2}\int_{\mathscr{S}} \mathbf{v}\,.\,\mathbf{vr} \times \mathbf{n}\, d\mathscr{S} = \frac{1}{2}\int_{\mathscr{S}_0} \mathbf{v}\,.\,\mathbf{vr} \times \mathbf{n}\, d\mathscr{S}_0 + \int_{\mathscr{V}'} \mathbf{r} \times (\nabla\mathbf{v})\,.\,\mathbf{v}\, d\mathscr{V}', \tag{33}$$

and, by (23),

$$\int_{\mathscr{S}_0} \mathbf{v}\,.\,\mathbf{vr} \times \mathbf{n}\, d\mathscr{S}_0 \to (\mathbf{r})_s \times \int_{\mathscr{S}_0} \mathbf{v}\,.\,\mathbf{vn}\, d\mathscr{S}_0 \to \frac{8\pi}{3} P_q D_s^q (\mathbf{r} \times \mathbf{v}')_s. \tag{34}$$

Hence, eliminating the volume integral between (32) and (33), we obtain

$$\int_{\mathscr{S}} \mathbf{r} \times [\tfrac{1}{2}\mathbf{v}\,.\,\mathbf{vn} - \mathbf{vv}\,.\,\mathbf{n}]\, d\mathscr{S} = -4\pi \sum_s P_q D_s^q (\mathbf{r} \times \mathbf{v}')_s. \tag{35}$$

Secondly, when $\mathbf{u} = \mathbf{V}$, a constant vector, (29) yields

$$\int_{\mathscr{S}} \mathbf{r} \times [\mathbf{v}.\mathbf{V}\mathbf{n} - \mathbf{v}\mathbf{V}.\mathbf{n}]\, d\mathscr{S} = -\mathbf{V} \times \int_{\mathscr{S}} \Phi \mathbf{n}\, d\mathscr{S} \tag{36}$$

since, by (9) and (26),

$$\int_{\mathscr{V}'} \mathbf{v} \times \mathbf{V}\, d\mathscr{V}' = -\mathbf{V} \times \int_{\mathscr{V}'} \nabla\Phi\, d\mathscr{V}' = -\mathbf{V} \times \left[\int_{\mathscr{S}} \Phi \mathbf{n}\, d\mathscr{S} - \sum_s \int_{\mathscr{S}_0} \Phi \mathbf{n}\, d\mathscr{S}_0\right] \to -\mathbf{V} \times \int_{\mathscr{S}} \Phi \mathbf{n}\, d\mathscr{S}. \tag{37}$$

Thirdly, letting $\mathbf{u} = \boldsymbol{\omega} \times \mathbf{r}$, we obtain from (29)

$$\int_{\mathscr{S}} \mathbf{r} \times [\mathbf{v}.\boldsymbol{\omega} \times \mathbf{r}\mathbf{n} - \mathbf{v}\boldsymbol{\omega} \times \mathbf{r}.\mathbf{n}]\, d\mathscr{S} = -\boldsymbol{\omega} \times \int_{\mathscr{S}} \Phi \mathbf{r} \times \mathbf{n}\, d\mathscr{S}, \tag{38}$$

which is derived by applying (30), the vector identity

$$\mathbf{r} \times (\mathbf{v} \times \mathbf{u}) + \mathbf{v} \times (\mathbf{u} \times \mathbf{r}) + \mathbf{u} \times (\mathbf{r} \times \mathbf{v}) = 0, \tag{39}$$

and the relation

$$\int_{\mathscr{V}'} \mathbf{r} \times \mathbf{v}\, d\mathscr{V}' = \int_{\mathscr{S}} \Phi \mathbf{r} \times \mathbf{n}\, d\mathscr{S} - \sum_s \int_{\mathscr{S}_0} \Phi \mathbf{r} \times \mathbf{n}\, d\mathscr{S} \to \int_{\mathscr{S}} \Phi \mathbf{r} \times \mathbf{n}\, d\mathscr{S}. \tag{40}$$

Of the above transformations, the three special forms (24), (25) and (28) will be used in the derivation of the force expression. Similarly, (35), (36), and (38) will be used for the moment.

3. Derivation of expression for the Lagally force

The hydrodynamic force acting on the body is given by (5) and (7) as

$$\mathbf{F} = \rho \int_{\mathscr{S}} \frac{\partial \Phi}{\partial t'} \mathbf{n}\, d\mathscr{S} + \rho \int_{\mathscr{S}} [\tfrac{1}{2}\mathbf{v}.\mathbf{v} - \mathbf{v}.\mathbf{V} - \mathbf{v}.\boldsymbol{\omega} \times \mathbf{r}]\mathbf{n}\, d\mathscr{S}. \tag{41}$$

Let us denote the first and second integrals in the right-hand side of (41) by $\mathbf{F}_1$ and $\mathbf{F}_2$, respectively. For a deformable surface, we may write

$$\frac{1}{\rho}\mathbf{F}_1 = \int_{\mathscr{S}} \frac{\partial \Phi}{\partial t'} \mathbf{n}\, d\mathscr{S} = \frac{d}{dt'} \int_{\mathscr{S}} \Phi \mathbf{n}\, d\mathscr{S} - \int_{\mathscr{S}} \mathbf{v}\mathbf{V}_d.\mathbf{n}\, d\mathscr{S}, \tag{42}$$

and by applying Green's reciprocal theorem and the relation (6) between absolute and relative time derivatives, we can rewrite the above equation as

$$\frac{1}{\rho}\mathbf{F}_1 = \frac{d}{dt}\left[\int_{\mathscr{S}} r \frac{\partial \Phi}{\partial n}\, d\mathscr{S} + \sum_s \int_{\mathscr{S}_0} \left(\Phi \mathbf{n} - \mathbf{r}\frac{\partial \Phi}{\partial n}\right) d\mathscr{S}_0\right] - \int_{\mathscr{S}} \mathbf{v}\mathbf{V}_d.\mathbf{n}\, d\mathscr{S} - \boldsymbol{\omega} \times \int_{\mathscr{S}} \Phi \mathbf{n}\, d\mathscr{S}. \tag{43}$$

For the first integral in the right-hand side of (43) we obtain, using (2),

$$\int_{\mathscr{S}} \mathbf{r} \frac{\partial \Phi}{\partial n}\, d\mathscr{S} = \int_{\mathscr{S}} \mathbf{r} \frac{\partial}{\partial n}(\Phi - \phi_d)\, d\mathscr{S} + \int_{\mathscr{S}} \mathbf{r} \frac{\partial \Phi_d}{\partial n}\, d\mathscr{S} = \int_{\mathscr{S}} \mathbf{r}(\mathbf{V}.\mathbf{n} + \boldsymbol{\omega}.\mathbf{r} \times \mathbf{n})\, d\mathscr{S} + \int_{\mathscr{S}} \mathbf{r}\mathbf{V}_d.\mathbf{n}\, d\mathscr{S}.$$

Furthermore, by applying the Gauss transformation, we obtain

$$\int_{\mathscr{S}} \mathbf{r}\frac{\partial \Phi}{\partial n}\,d\mathscr{S} = \mathbf{V}_c\mathscr{V} + \int_{\mathscr{S}} \mathbf{r}\mathbf{V}_d.\mathbf{n}\,d\mathscr{S}, \tag{44}$$

where $\mathbf{V}_c$ is the velocity of the volume centroid.

For the second integral in the right-hand side of (43), we write

$$\int_{\mathscr{S}_0}\left(\Phi\mathbf{n} - \mathbf{r}\frac{\partial\Phi}{\partial n}\right)d\mathscr{S}_0 = -\int_{\mathscr{S}_0}\mathbf{r}(\mathbf{v}'+\mathbf{v}_q).\mathbf{n}\,d\mathscr{S}_0 \to -4\pi P_q D_s^q(\mathbf{r})_s \tag{45}$$

because of (26). We also note that $D_s^q(\mathbf{r})_s$ vanishes for $q \geqslant 2$. Substituting (44) and (45) into (43), we now obtain

$$\frac{1}{\rho}\mathbf{F}_1 = \frac{d}{dt}\left[\mathbf{V}_c\mathscr{V} - 4\pi\sum_s P_q D_s^q(\mathbf{r})_s + \int_{\mathscr{S}}\mathbf{r}\mathbf{V}_d.\mathbf{n}\,d\mathscr{S}\right] - \int_{\mathscr{S}}\mathbf{v}\mathbf{V}_d.\mathbf{n}\,d\mathscr{S} - \boldsymbol{\omega}\times\int_{\mathscr{S}}\Phi\mathbf{n}\,d\mathscr{S}. \tag{46}$$

Next applying the transformations (24), (25) and (28), together with the boundary condition (2), we obtain the following for $\mathbf{F}_2$:

$$\frac{1}{\rho}\mathbf{F}_2 = \int_{\mathscr{S}}\mathbf{v}\mathbf{V}_d.\mathbf{n}\,d\mathscr{S} + \boldsymbol{\omega}\times\int_{\mathscr{S}}\Phi\mathbf{n}\,d\mathscr{S} - 4\pi\sum_s P_q D_s^q(\mathbf{v}')_s. \tag{47}$$

We also note that, by the Reynolds transport theorem,

$$\int_{\mathscr{S}}\mathbf{r}\mathbf{V}_d.\mathbf{n}\,d\mathscr{S} = \frac{d}{dt}(\mathbf{r}_c\mathscr{V}). \tag{48}$$

Hence the final expression for $\mathbf{F}$, obtained from (47) and (48), is

$$\frac{1}{\rho}\mathbf{F} = \frac{d}{dt}\left[\mathbf{V}_c\mathscr{V} - 4\pi\sum_s P_q D_s^q(\mathbf{r})_s + \frac{d}{dt}(\mathbf{r}_c\mathscr{V})\right] - 4\pi\sum_s P_q D_s^q(\mathbf{v}')_s \tag{49}$$

as was stated in (10). For the case of a rigid surface, $\mathbf{V}_d.\mathbf{n} = 0$, and for multipoles of order 0 and 1 ($q \leqslant 1$), (49) reduces to the corresponding expression given in Landweber & Yih (1956). The above expression is useful when the normal velocity $\mathbf{V}_d.\mathbf{n}$ of the deformable surface is prescribed. For the case where the deformation velocity is associated with a potential Φ_d, an alternative form of (49), obtained by modifying (42), is

$$\frac{1}{\rho}\mathbf{F} = \frac{d}{dt}\left[\mathbf{V}_c\mathscr{V} - 4\pi\sum_s^{(d)} P_q D_s^q(\mathbf{r})_s + \int_{\mathscr{S}}\Phi_d\mathbf{n}\,d\mathscr{S}\right] - 4\pi\sum_s P_q D_s^q(\mathbf{v}')_s, \tag{50}$$

where the symbol $\sum_s^{(d)}$ denotes summation over all singularities except those associated with the deformation potential ϕ_d.

In applying the formula for the force, (49) or (50), the contributions to $(v')_s$ from singularities at other points within the volume $\mathscr{V}$ may be neglected since, as is shown in the other referenced treatments of the Lagally theorem, such terms cancel when summed over all the internal singularities.

4. Derivation of expression for the Lagally moment

The hydrodynamical moment is given by (5) and (7) as

$$\mathbf{M} = \rho\int_{\mathscr{S}} \frac{\partial\Phi}{\partial t'}\,\mathbf{r}\times\mathbf{n}\,d\mathscr{S} + \rho\int_{\mathscr{S}} [\tfrac{1}{2}\mathbf{v}.\mathbf{v} - \mathbf{v}.\mathbf{V} - \mathbf{v}.\boldsymbol{\omega}\times\mathbf{r}](\mathbf{r}\times\mathbf{n})\,d\mathscr{S}. \qquad (51)$$

We denote the first integral on the right-hand side of (51) by $\mathbf{M}_1$ and, using (6), we express it as

$$\frac{1}{\rho}\mathbf{M}_1 = \int_{\mathscr{S}} \frac{\partial\Phi}{\partial t'}\,\mathbf{r}\times\mathbf{n}\,d\mathscr{S} = \frac{d}{dt}\int_{\mathscr{S}} \Phi\mathbf{r}\times\mathbf{n}\,d\mathscr{S} - \boldsymbol{\omega}\times\int_{\mathscr{S}}\Phi\mathbf{r}\times\mathbf{n}\,d\mathscr{S} + \int_{\mathscr{S}} \mathbf{v}\times\mathbf{r}\mathbf{V}_d.\mathbf{n}\,d\mathscr{S}. \qquad (52)$$

Furthermore, by (1) and Green's reciprocal formula, applied to the region exterior to $\mathscr{S}$ for ϕ_i, ϕ_{3+i} and ϕ_d, and to the interior region for ϕ_0, we obtain

$$\int_{\mathscr{S}} \Phi(\mathbf{r}\times\mathbf{n})_j\,d\mathscr{S} = \sum_{i=1}^{3}\int_{\mathscr{S}} (V_i\phi_i + \omega_i\phi_{3+i} + \phi_0 + \phi_d)\frac{\partial\phi_{3+j}}{\partial n}\,d\mathscr{S}$$
$$= \sum_{i=1}^{3}\int_{\mathscr{S}}\left(V_i\frac{\partial x_i}{\partial n} + \omega_i\frac{\partial\phi_{3+i}}{\partial n} + \frac{\partial\phi_d}{\partial n}\right)\phi_{3+j}\,d\mathscr{S} + \sum_s\int_{\mathscr{S}_0}\left(\phi_0\frac{\partial\phi_{3+j}}{\partial n} - \phi_{3+j}\frac{\partial\phi_0}{\partial n}\right)d\mathscr{S}_0. \qquad (53)$$

Put $\phi_0 = \phi_0' + \phi_{0q}$ and $\phi_{3+j} = \phi_{3+j}' + \phi_{3+j,q'}$ in the last integral. Then we have, by the reciprocal formula,

$$\int_{\mathscr{S}_0}\left(\phi_{0q}\frac{\partial\phi_{3+j,q'}}{\partial n} - \phi_{3+j,q'}\frac{\partial\phi_{0q}}{\partial n}\right)d\mathscr{S}_0 = 0.$$

We also have

$$\int_{\mathscr{S}_0}\phi_{3+j}'\frac{\partial\phi_{0q}}{\partial n}\,d\mathscr{S}_0 \to 4\pi P_{0q}\,D_s^q(\phi_{3+j}')_s, \quad \int_{\mathscr{S}_0}\phi_0'\frac{\partial\phi_{3+j,q'}}{\partial n}\,d\mathscr{S}_0 \to 4\pi P_{3+j,q'}\,D_s^{q'}(\phi_0')_s$$

and

$$\int_{\mathscr{S}} V_i\frac{\partial x_i}{\partial n}\,\Phi_{3+j}\,d\mathscr{S} = \int_{\mathscr{S}} V_i x_i(\mathbf{r}\times\mathbf{n})_j\,d\mathscr{S} + \sum_s\int_{\mathscr{S}_0}\left(\frac{\partial x_i}{\partial n}\phi_{3+j} - x_i\frac{\partial\phi_{3+j}}{\partial n}\right)V_i\,d\mathscr{S}_0$$
$$\to (\mathbf{r}_c\times\mathbf{V})_j\mathscr{V} - 4\pi\sum_s P_{3+j,q'}D_s^{q'}(\mathbf{V}.\mathbf{r})_s. \qquad (54)$$

Then, introducing the added-mass coefficients in (53), we obtain

$$\int_{\mathscr{S}}\Phi(\mathbf{r}\times\mathbf{n})_j\,d\mathscr{S} = (\mathbf{r}_c\times\mathbf{V})_j\mathscr{V} - \omega_i A_{3+i,3+j} - 4\pi\sum_s[P_{3+j,q'}D_s^{q'}(\mathbf{V}.\mathbf{r}-\phi_0')_s + P_{0q}D_s^q(\phi_{3+j}')_s] + \int_{\mathscr{S}}\phi_{3+j}\mathbf{V}_d.\mathbf{n}\,d\mathscr{S}. \qquad (55)$$

Next, applying the transformations (35), (36) and (38) to the second term on the right-hand side of (52) gives, after making use of the boundary condition (2),

$$\frac{1}{\rho}\mathbf{M}_2 = \int_{\mathscr{S}}[\tfrac{1}{2}\mathbf{v}.\mathbf{v} - \mathbf{v}.\mathbf{V} - \mathbf{v}.\boldsymbol{\omega}\times\mathbf{r}](\mathbf{r}\times\mathbf{n})\,d\mathscr{S} = \mathbf{V}\times\int_{\mathscr{S}}\Phi\mathbf{n}\,d\mathscr{S}$$
$$+\boldsymbol{\omega}\times\int_{\mathscr{S}}\Phi\mathbf{r}\times\mathbf{n}\,d\mathscr{S} + \int_{\mathscr{S}}\mathbf{r}\times\mathbf{v}\mathbf{V}_d.\mathbf{n}\,d\mathscr{S} - 4\pi\sum_s P_q D_s^q(\mathbf{r}\times\mathbf{v}')_s. \qquad (56)$$

However, as shown in (44) and (45),

$$\int_{\mathscr{S}} \Phi \mathbf{n}\, d\mathscr{S} = \mathbf{V}_c \mathscr{V} - 4\pi \sum_s P_q D_s^q(\mathbf{r})_s + \int_{\mathscr{S}} r\mathbf{V}_d . \mathbf{n}\, d\mathscr{S}. \tag{57}$$

Hence, from (52) to (57), and (48),

$$\frac{1}{\rho}\mathbf{M}_j = \left[\mathbf{r}_c \times \frac{d}{dt}(\mathbf{V}\mathscr{V})\right]_j - \frac{d}{dt}\left\{\omega_i A_{3+j,3+i} + 4\pi \sum_s [P_{3+j,q'} D_s^q(\mathbf{V}.\mathbf{r} - \phi_0')_s + P_{0q} D_s^q(\phi_{3+j}')_s] + \int_{\mathscr{S}} \phi_{3+j} \mathbf{V}_d . \mathbf{n}\, d\mathscr{S}\right\} - 4\pi \sum_s P_q D_s^q[\mathbf{r} \times (\mathbf{v}' - \mathbf{V})]_{js} + \left[\mathbf{V} \times \frac{d}{dt}(\mathbf{r}_c \mathscr{V})\right]_j \tag{58}$$

as we wished to show.

An alternative expression for the moment, when the deformation velocity is associated with a potential ϕ_d, is

$$\frac{1}{\rho}\mathbf{M}_j = \left[\mathbf{r}_c \times \frac{d}{dt}(\mathbf{V}\mathscr{V})\right]_j - \frac{d}{dt}\{\omega_i A_{3+j,3+i} - A_{d,3+j} + 4\pi \sum_s [P_{3+j,q'} D_s^{q'}(\mathbf{V}.\mathbf{r} - \phi_0')_s + P_{0q} D_s^q(\phi_{3+j}')_s]\} - 4\pi \sum_s^{(d)} P_q D_s^q[\mathbf{r} \times (\mathbf{v}' - \mathbf{V})]_{js} - 4\pi \sum_s P_{d,q} D_s^q(\mathbf{r} \times \mathbf{v}')_s - e_{ijk} V_k A_{di}, \tag{59}$$

where e_{ijk} is the permutation tensor and A_{di} and $A_{d,j+3}$ are added masses associated with the interference between the translatory and the rotational motion of the body and the motion induced by its deformation, i.e.

$$A_{di} = -\int_{\mathscr{S}} \phi_d n_i\, d\mathscr{S}, \quad A_{d,3+j} = -\int_{\mathscr{S}} \phi_d(\mathbf{r} \times \mathbf{n})_j\, d\mathscr{S}. \tag{60}$$

For the case of a rigid body where the highest degree of the multipoles is 1, i.e. $q = 0, 1$, the singularity distribution consists of sources and doublets and the expression for the moment given in (58) reduces to the equivalent expression given in Landweber & Yih (1956). For a rigid body in steady flow represented by multipoles of arbitrary order, (10) and (11) are identical with the corresponding equations given in Landweber (1967). For the case of a deformable surface moving in an unbounded fluid, the generalized Lagally expressions reduce to the form given in Averbukh (1973) in terms of added-mass coefficients. Equations (10) and (11) are more general in the sense that they apply to a general unsteady motion of deformable or permeable bodies which are represented by multipoles of arbitrary order.

So far the analysis has been carried out for a discrete distribution of multipoles. When continuous distributions of multipoles exist within the volume $\mathscr{V}$, the expressions (10) and (11) for **F** and **M** need to be modified only by replacing the sums over the multipole terms by integrals over their distributions. This will be illustrated in the examples that follow.

5. Manoeuvring of a deformable ellipsoid

In this section we consider the forces and moments acting on a deformable tri-axial ellipsoid moving unsteadily in an otherwise undisturbed medium, in a motion having six degrees of freedom. The instantaneous equation of the ellipsoidal surface is

$$x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 = 1, \tag{61}$$

where the time-dependent major axes are arranged such that $a_1 > a_2 > a_3$. It is assumed that, during its course of deformation, the ellipsoid remains similar to itself, that is to say

$$\frac{1}{a_i}\frac{da_i}{dt'} = \lambda(t), \quad i = 1, 2, 3. \tag{62}$$

The general motion of the ellipsoid consists of a translatory velocity vector $\mathbf{V}(V_1, V_2, V_3)$ and angular velocity $\boldsymbol{\omega}(\omega_1, \omega_2, \omega_3)$ about the major axes of the ellipsoid. The six velocity components are assumed to be time dependent.

In solving potential flow problems involving ellipsoidal boundaries, it is convenient to employ orthogonal ellipsoidal co-ordinates (η, μ, ν) in the manner defined by Hobson (1955, p. 454). A normal solution of the Laplace equation in ellipsoidal co-ordinates (ellipsoidal harmonic) which vanishes at infinity may be written as $\mathscr{F}_j^i(\eta)\,\mathscr{E}_j^i(\mu)\,\mathscr{E}_j^i(\nu)$, where $\mathscr{E}_j^i$ and $\mathscr{F}_j^i$ are the Lamé functions of the first and second kind respectively. Also i and j are integers such that $i \leqslant 2j+1$.

The three Kirchhoff potentials associated with the translatory motion of the ellipsoid are given by Miloh (1973) as

$$\phi_i(\eta, \mu, \nu) = KC_1^i[|a_{i+1}^2 - a_{i+2}^2|]^{\frac{1}{2}}\,\mathscr{F}_1^i(\eta)\,\mathscr{E}_1^i(\mu)\,\mathscr{E}_1^i(\nu), \tag{63}$$

where $a_{i+3} = a_i$,

$$K = [(a_1^2 - a_2^2)(a_2^2 - a_3^2)(a_1^2 - a_3^2)]^{-\frac{1}{2}}, \tag{64}$$

and, in general,

$$C_j^i = \left[\frac{d}{d\eta}\mathscr{E}_j^i(\eta)\right] \Big/ \left[\frac{d}{d\eta}\mathscr{F}_j^i(\eta)\right]_{\eta = a_1}. \tag{65}$$

Similarly, the three Kirchhoff potentials associated with the rotational motion of the ellipsoid are given by Miloh (1973) as

$$\phi_{3+i}(\eta, \mu, \nu) = KC_2^{6-i}\frac{[|a_{i+1}^2 - a_{i+2}^2|]^{\frac{1}{2}}}{a_{i+1}^2 + a_{i+2}^2}\,\mathscr{F}_2^{6-i}(\eta)\,\mathscr{E}_2^{6-i}(\mu)\,\mathscr{E}_2^{6-i}(\nu). \tag{66}$$

The potential function associated with the deformation of the ellipsoid may be expressed as

$$\phi_d(\eta, \mu, \nu) = -B\mathscr{F}_0(\eta)\,\mathscr{E}_0(\mu)\,\mathscr{E}_0(\nu), \tag{67}$$

where B is a coefficient to be determined from condition (62). The normal velocity on the surface of the ellipsoid induced by the deformation potential ϕ_d is

$$\mathbf{V}_d \cdot \mathbf{n} = \frac{B}{a_1 a_2 a_3}\left[\frac{x_1^2}{a_1^4} + \frac{x_2^2}{a_2^4} + \frac{x_3^2}{a_3^4}\right]^{-\frac{1}{2}}, \tag{68}$$

which together with (62) implies that $B = \lambda a_1 a_2 a_3$.

The ultimate image singularity system of the six Kirchhoff potentials and the deformation potential within the ellipsoid (Miloh 1974) consists of a distribution of sources of strength

$$M(x_1, x_2, 0) = -\frac{1}{2\pi}(a_1^2-a_3^2)^{-\frac{1}{2}}(a_2^2-a_3^2)^{-\frac{1}{2}}\left(1-\frac{x_1^2}{a_1^2-a_3^2}-\frac{x_2^2}{a_2^2-a_3^2}\right)^{-\frac{1}{2}}$$
$$\times\left[\frac{3C_1^1V_1x_1}{a_1^2-a_3^2}+\frac{3C_1^2V_2x_2}{a_2^2-a_3^2}+\frac{5C_2^3\omega_3(a_1^2-a_2^2)x_1x_2}{(a_1^2-a_3^2)(a_1^2+a_2^2)(a_2^2-a_3^2)}-\lambda a_1a_2a_3\right] \quad (69)$$

and normal doublets (Miloh 1974)

$$N(x_1, x_2, 0) = -\frac{1}{2\pi}(a_1^2-a_3^2)^{-\frac{1}{2}}(a_2^2-a_3^2)^{-\frac{1}{2}}\left(1-\frac{x_1^2}{a_1^2-a_3^2}-\frac{x_2^2}{a_2^2-a_3^2}\right)^{\frac{1}{2}}$$
$$\times\left[3C_1^3V_3+\frac{5C_2^5\omega_1x_2}{a_2^2+a_3^2}-\frac{5C_2^4\omega_2x_1}{a_1^2+a_3^2}\right] \quad (70)$$

over the 'fundamental ellipse' given by

$$\frac{x_1^2}{a_1^2-a_3^2}+\frac{x_2^2}{a_2^2-a_3^2} = 1, \quad x_3 = 0. \quad (71)$$

Since the image singularities system consists of only sources and normal doublets distributed continuously over part of the $x_3 = 0$ plane, the hydrodynamic force [see (10)] is given by

$$\frac{1}{\rho}\mathbf{F} = \frac{d}{dt}\left[\mathbf{V}_c\mathscr{V}-4\pi\int_{\bar{\mathscr{S}}}\left(M\mathbf{r}+N\frac{\partial\mathbf{r}}{\partial x_3}\right)d\bar{\mathscr{S}}+\int_{\mathscr{S}}\mathbf{r}\mathbf{V}_d.\mathbf{n}\,d\mathscr{S}\right], \quad (72)$$

where $\bar{\mathscr{S}}$ denotes the ellipse given in (71) and $\mathscr{S}$ the ellipsoid in (61). Substituting (68), (69), and (70) into (72) yields

$$\frac{1}{\rho}\mathbf{F}_i = \frac{d}{dt}[V_i\mathscr{V}+4\pi V_iC_1^i] \quad (73)$$

since the integral over $\mathscr{S}$ in (72) vanishes because of symmetry. Since the three longitudinal added-mass coefficients A_{ii} are related to the three coefficients C_1^i by the relations (Miloh 1973)

$$A_{ii} = -(\mathscr{V}+4\pi C_1^i), \quad (74)$$

(75) then reduces to

$$\frac{1}{\rho}\mathbf{F}_i = -\frac{d}{dt}(A_{ii}V_i). \quad (75)$$

By transforming the absolute time derivative in (75) into a time derivative relative to the moving co-ordinate system (6), we get

$$\frac{1}{\rho}\mathbf{F}_i = -A_{ii}\frac{dV_i}{dt'}+e_{ijk}A_{jj}V_j\omega_k-\lambda V_i\sum_{j=1}^{3}a_j\frac{\partial A_{ii}}{\partial a_j}. \quad (76)$$

For the case of a non-deformable (rigid) body, we have $\lambda = 0$ and (76) reduces to the well-known expression given in Kochin, Kibel & Roze (1965, p. 401). For the case of a spherical body in the same motion we have $A_{ii} = \frac{1}{2}\mathscr{V}$ and (76) yields

$$\frac{1}{\rho}\mathbf{F}_i = -\frac{1}{2}\mathscr{V}\left(\frac{dV_i}{dt'}+3\lambda V_i-e_{ijk}V_j\omega_k\right). \quad (77)$$

Next we calculate the hydrodynamic moment experienced by the ellipsoid. For the present case, there are no external boundaries or flow-producing mechanisms.

Therefore both ϕ_0 and $\mathbf{v}_0$ vanish. Since the origin of the co-ordinate system coincides with the volume centroid, we have also $\mathbf{r}_c = 0$. Equation (11) then yields

$$\frac{1}{\rho}\mathbf{M}_i = -\frac{d}{dt}\left\{\omega_j A_{3+i,\,3+j} + 4\pi\int_{\bar{\mathscr{S}}}\left[M_{3+i}\,\mathbf{V}\,.\,\mathbf{r} + N_{3+i}\frac{\partial}{\partial x_3}(\mathbf{V}\,.\,\mathbf{r})\right]d\bar{\mathscr{S}} + \int_{\mathscr{S}}\phi_{3+i}\,\mathbf{V}_d\,.\,\mathbf{n}\,d\mathscr{S}\right\}$$
$$+4\pi\int_{\bar{\mathscr{S}}}\left[M(\mathbf{r}\times\mathbf{V})_i + N\frac{\partial}{\partial x_3}(\mathbf{r}\times\mathbf{V})_i\right]d\bar{\mathscr{S}} + \int_{\mathscr{S}}\ldots + \int_{\mathscr{S}}(\mathbf{V}\times\mathbf{r})_i\,\mathbf{V}_d\,.\,\mathbf{n}\,d\mathscr{S}, \quad (78)$$

where M_{3+i} and N_{3+i} are, respectively, the source and doublet distributions associated with ϕ_{3+i}. Because of the symmetrical properties of the ellipsoid, the integrals over $\mathscr{S}$ in (78) vanish and the above expression may be further reduced to

$$\frac{1}{\rho}\mathbf{M}_i = -\frac{d}{dt}(\omega_i A_{3+i,3+i}) + 4\pi\int_{\bar{\mathscr{S}}}\left[M(\mathbf{r}\times\mathbf{V})_i + N\frac{\partial}{\partial x_3}(\mathbf{r}\times\mathbf{V})_i\right]d\mathscr{S}. \quad (79)$$

In a similar manner to (76), we obtain the following expression for the hydrodynamic moment acting on the ellipsoid:

$$\frac{1}{\rho}\mathbf{M}_i = -A_{3+i,3+i}\frac{d\omega_i}{dt'} + e_{ijk}[V_j V_k A_{jj} + \omega_j\omega_k A_{3+j,3+j}] - \lambda\omega_i\sum_{j=1}^{3} a_j\frac{\partial A_{3+i,3+i}}{\partial a_j}. \quad (80)$$

Again the above expression reduces to that for a rigid surface (Kochin *et al.* 1965, p. 401) and, as expected, vanishes for a spherical surface. The six added-mass coefficients of the ellipsoid are geometrical parameters and may be expressed in terms of tabulated elliptic integrals (Munk 1934, p. 301; Miloh 1973).

6. Force on an expanding sphere in axisymmetric flow

As a second example, the force on a translating sphere with a time-dependent radius will be presented. The undisturbed flow field is assumed to be symmetric with respect to the x axis along which the sphere is translating with velocity V. The undisturbed velocity potential (without the sphere) is expressed in terms of interior spherical harmonics as $\sum_{n=1}^{\infty} A_n R^n P_n(\mu)$, where (R,θ) are axisymmetrical spherical co-ordinates, $\mu = \cos\theta$, $P_n(\mu)$ denotes the Legendre polynomials and A_n are given time-dependent coefficients. For convenience, the value of the undisturbed potential at the origin is taken to be zero. After introducing the sphere into the flow field, the velocity potential may be written as

$$\phi(R,\mu) = \sum_{n=0}^{\infty} B_n R^{-(n+1)}P_n(\mu) + \sum_{n=1}^{\infty} A_n R^n P_n(\mu). \quad (81)$$

At a certain instant, let the sphere radius be a and let this spherical surface expand with velocity $\dot{a}$ (in the radial direction). For these boundary conditions, (81) yields

$$B_n = \frac{n}{n+1}a^{2n+1}A_n, \quad n \geqslant 1,$$

and

$$B_0 = -\dot{a}a^2. \quad (82)$$

Following Hobson (1955, p. 133) we have

$$\sum_{n=0}^{\infty} B_n R^{-(n+1)} P_n(\mu) = \sum_{n=0}^{\infty} B_n \frac{(-1)^n}{n!} \frac{\partial^n}{\partial x^n}\left(\frac{1}{R}\right), \tag{83}$$

which implies that the image of the exterior disturbance potential is given by an infinite series of multipoles in the x direction, lying at the origin. The strength of the nth-order multipole is given by

$$P_n = -\frac{1}{n!} B_n. \tag{84}$$

The sphere will experience a force in the x direction which may be computed from the generalized Lagally expression (10) to yield

$$\frac{1}{\rho} F_1 = \frac{d}{dt}\left[V\mathscr{V} + 4\pi \sum_{n=0}^{\infty} \frac{1}{n!} B_n \frac{\partial^n}{\partial x^n}(x)\right] + 4\pi \sum_{n=0}^{\infty} \frac{1}{n!} B_n \frac{\partial^{n+1}}{\partial x^{n+1}}\left[\sum_{m=1}^{\infty} A_m R^m P_m(\mu)\right], \tag{85}$$

which is to be evaluated at $R = 0$. Making use of the relation

$$\frac{\partial^n}{\partial x^n}[R^m P_m(\mu)] = P_m(0)\frac{\partial^n x^m}{\partial x^n} = \begin{Bmatrix} 0, & m < n \\ n!, & m = n \\ 0, & m > n \end{Bmatrix}, \quad R = 0, \tag{86}$$

we get

$$\frac{1}{\rho} F_1 = \left[\mathscr{V}\frac{dV}{dt} + V\frac{d\mathscr{V}}{dt} + 4\pi\frac{dB_1}{dt}\right] + 4\pi \sum_{n=1}^{\infty} (n+1) B_n A_{n+1} \tag{87}$$

or, applying (82),

$$\frac{1}{\rho\mathscr{V}} F_1 = \frac{dV}{dt} + \frac{3}{2}\frac{dA_1}{dt} + 3\frac{\dot{a}}{a}(V + \tfrac{3}{2}A_1) + 3\sum_{n=1}^{\infty} n a^{2n-2} A_n A_{n+1}. \tag{88}$$

For the case where the sphere is translating in an unbounded medium otherwise at rest, we have $A_1 = -V$ and $A_n = 0$ for $n \neq 1$, and (88) is reduced to the previously derived expression (77) for the force acting on a deformable, non-rotating sphere. In a similar manner we obtain that the force on a stationary, deformable sphere in a uniform flow $U(t)$ is given by

$$\frac{1}{\rho\mathscr{V}} F_1 = \frac{3}{2}\left(\frac{dU}{dt} + 3U\frac{\dot{a}}{a}\right). \tag{89}$$

A rather interesting result may be easily derived from (89). A stationary sphere in a uniform unsteady stream will experience no force when the stream velocity and the sphere radius vary with time such that the product Ua^3 is constant. Similarly, when the sphere is moving with velocity V in an otherwise stationary medium, the hydrodynamic force experienced by the sphere also vanishes when Va^3 is constant.

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